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HILL CIPHER, STREAM CIPHERS AND THE ONE TIME PAD

TTM4135 - Lecture 3

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12.01.2026

Motivation

- The Hill Cipher is a mathematically defined encryption scheme
- The Hill Cipher illustrates the weakness of linearity in cipher design
- Stream ciphers are constructed from (pseudo-)random number generators where a keystrem is xor-ed with the message
- The One Time Pad is an unbreakable stream cipher

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Hill cipher

- Lester Hill, an American mathematician, published his cipher in 1929.
- The Hill cipher is an example of a *polygram cipher* (also called *polygraphic cipher*). This is a simple substitution cipher on an extended alphabet consisting of multiple characters. The simplest example is digram substitution in which the alphabet consists of all pairs of characters.
- The major weakness of the Hill cipher is that it is linear. This makes known plaintext attacks easy.

Definition of Hill cipher

The Hill cipher performs a linear transformation on d plaintext characters to get d ciphertext characters.

- Encryption involves multiplying a $d \times d$ matrix $\mathbf{K}$ by a plaintext block $\mathbf{P}$.
- Decryption involves multiplying the matrix $\mathbf{K}^{-1}$ by the block of the ciphertext $\mathbf{C}$.

$$\begin{aligned} \text{Encryption:} \quad & \mathbf{C} = \mathbf{K}\mathbf{P} \\ \text{Decryption:} \quad & \mathbf{P} = \mathbf{K}^{-1}\mathbf{C} \end{aligned}$$

Encryption example

- We choose $d = 2$ so encryption takes digrams as input and output blocks
- Write each plaintext pair as a column vector and encode them as numbers
- Suppose the first pair for encryption is (EG). Then since E=4 and G=6 in our encoding this is represented as $\begin{pmatrix} 4 \\ 6 \end{pmatrix}$
- If there are insufficient letters to fill a block then it must be padded. This can be done with an uncommon letter such as Z
- In these examples the space character is omitted and all computations take place modulo 26

Encrypting and decrypting

$$d = 2, \quad \mathbf{K} = \begin{pmatrix} 4 & 5 \\ 1 & 7 \end{pmatrix}, \quad \mathbf{K}^{-1} = \begin{pmatrix} 15 & 19 \\ 9 & 16 \end{pmatrix}$$

$$\textit{Plaintext} : (BC) \rightarrow \mathbf{P} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\textit{Encryption} : \mathbf{C} = \mathbf{KP} = \begin{pmatrix} 4 & 5 \\ 1 & 7 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 14 \\ 15 \end{pmatrix} \rightarrow (OP)$$

$$\textit{Decryption} : \mathbf{P} = \mathbf{K}^{-1}\mathbf{C} = \begin{pmatrix} 15 & 19 \\ 9 & 16 \end{pmatrix} \begin{pmatrix} 14 \\ 15 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Cryptanalysis of Hill cipher

- Known plaintext attack is possible given d matching blocks.
- Suppose we are given blocks (column vectors) $\mathbf{P}_i, \mathbf{C}_i$ for $i = 0, 1, .., d-1$.

1. Let $\mathbf{C} = [\mathbf{C}_0\, \mathbf{C}_1\, \ldots\, \mathbf{C}_{d-1}]$. Let $\mathbf{P} = [\mathbf{P}_0\, \mathbf{P}_1\, \ldots\, \mathbf{P}_{d-1}]$.
2. Solve $\mathbf{C} = \mathbf{KP}$ for $\mathbf{K}$. Compute $\mathbf{P} = \mathbf{K}^{-1}\mathbf{C}$.

Cryptanalysis example

- Suppose that we know $d = 2$.
- Ciphertext: ZIKPWIXPTFUTVPVRQBUTVPJLKB
- Known plaintext is first two blocks (digrams): THER

Step 1 - Encode plaintext and ciphertext

$$\mathbf{P}_0 = (TH) = \begin{pmatrix} 19 \\ 7 \end{pmatrix}, \ \mathbf{P}_1 = (ER) = \begin{pmatrix} 4 \\ 17 \end{pmatrix}$$

$$\mathbf{C}_0 = (ZI) = \begin{pmatrix} 25 \\ 8 \end{pmatrix}, \ \mathbf{C}_1 = (KP) = \begin{pmatrix} 10 \\ 15 \end{pmatrix}$$

$$\rightarrow \mathbf{P} = [\mathbf{P}_0 \, \mathbf{P}_1] = \begin{pmatrix} 19 & 4 \\ 7 & 17 \end{pmatrix}$$

$$\rightarrow \mathbf{C} = [\mathbf{C}_0 \, \mathbf{C}_1] = \begin{pmatrix} 25 & 10 \\ 8 & 15 \end{pmatrix}$$

Step 2 - Recover encryption matrix K

We have $\mathbf{C} = \mathbf{KP}$. Let $\mathbf{K} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then:

$$\begin{pmatrix} 25 & 10 \\ 8 & 15 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 19 & 4 \\ 7 & 17 \end{pmatrix}.$$

So

$$\begin{aligned} \begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \begin{pmatrix} 25 & 10 \\ 8 & 15 \end{pmatrix} \begin{pmatrix} 19 & 4 \\ 7 & 17 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 25 & 10 \\ 8 & 15 \end{pmatrix} \begin{pmatrix} 25 & 14 \\ 5 & 5 \end{pmatrix} \\ &= \begin{pmatrix} 25 & 10 \\ 15 & 5 \end{pmatrix} \end{aligned}$$

Step 3 - Compute K^{-1} and decrypt ciphertext

$$\rightarrow \mathbf{K} = \begin{pmatrix} 25 & 10 \\ 15 & 5 \end{pmatrix} \rightarrow \cdots \rightarrow \mathbf{K}^{-1} = \begin{pmatrix} 5 & 16 \\ 11 & 25 \end{pmatrix}$$

$$\begin{aligned} \mathbf{C} &= \begin{pmatrix} W & X & T & U & T & \dots \\ I & P & F & T & V & \dots \end{pmatrix} \\ &= \begin{pmatrix} 22 & 23 & 17 & 20 & 17 & \dots \\ 8 & 15 & 5 & 17 & 21 & \dots \end{pmatrix} \end{aligned}$$

$$\mathbf{P} = \mathbf{K}^{-1}\mathbf{C}$$

Plaintext: THEREARETWOTHINGSTOTHINKOF

Notes on cryptanalysis of Hill cipher

- In known plaintext attacks the equations may not be fully determined. In this case Step 2 will fail because the matrix will not be invertible. Further plaintext/ciphertext character can be examined.
- Ciphertext only attack follows known plaintext attack with the added task of finding probable blocks of matching plaintext and ciphertext. For example, when $d = 2$ the frequency distribution of non-overlapping pairs of ciphertext characters can be compared with the distribution of pairs of plaintext characters.

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Stream ciphers

- Stream ciphers are characterized by the generation of a *keystream* of any required length
- Each element of the keystream is used successively to encrypt one or more ciphertext characters
- Stream ciphers are usually symmetric key ciphers: sender and receiver share the same key and can generate the same keystream given the same initialization value
- The keystream must have good randomness properties

Synchronous stream ciphers

- In the simplest kind of stream cipher the keystream is generated *independently* of the plaintext. In this case the cipher is called a *synchronous* stream cipher.
- Both sender and receiver need to generate the same keystream and synchronize on its usage
- The Vigenère cipher can be seen as a (periodic) synchronous stream cipher where each shift is defined by a key letter
- Later we will see how to use modern block ciphers to generate a keystream

Binary synchronous stream cipher

For each time interval t each of the following are defined:

- a binary sequence $s(t)$ called the *keystream*;
- a binary plaintext $p(t)$;
- a binary ciphertext $c(t)$.

$$\begin{aligned} &\text{Encryption:} \quad c(t) = p(t) \oplus s(t) \\ &\text{Decryption:} \quad p(t) = c(t) \oplus s(t) \end{aligned}$$

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One Time Pad

- Often attributed to Vernam who made a One Time Pad machine using teletype machinery in 1917. Earlier historical uses are known.
- The key is a truly random sequence of characters, all of them independently generated
- Each character in the key is used *one time* only
- The alphabet can be of any length, but usually is either a natural language alphabet or simply the binary alphabet $\{0, 1\}$.
- The binary One Time Pad is a binary synchronous stream cipher.
- The One Time Pad provides perfect secrecy.

Shannon's definition of perfect secrecy

- To define perfect secrecy, consider a cipher with message set $\{M_1, M_2, \ldots, M_k\}$ and ciphertext set $\{C_1, C_2, \ldots, C_l\}$.
- Then $\Pr(M_i|C_j)$ is the probability that message M_i was encrypted given that ciphertext C_j was observed.
- Note that in most cases the messages M_i will *not* be equally likely.
- We say that the cipher achieves *perfect secrecy* if for all messages M_i and ciphertexts C_j we have

$$\Pr(M_i|C_j) = \Pr(M_i)$$

One Time Pad using Roman alphabet

- Plaintext characters: $p_1, \ldots, p_r$
- Ciphertext characters: $c_1, \ldots, c_r$
- Keystream: random characters $k_1, \ldots, k_r$
- Encryption:

$$c_i = (p_i + k_i) \bmod 26$$

- Decryption:

$$p_i = (c_i - k_i) \bmod 26$$

- The resulting ciphertext is a modulo 26 addition of the plaintext and keystream sequences.

Why the One Time Pad provides perfect secrecy

- Suppose a particular ciphertext C_j is observed.
- Any message could have been sent depending on the choice of key.
- The probability that message M_i was sent given that C_j is observed is the probability that M_i is chosen, weighted by the probability that the right key was chosen.
- Since each key is chosen with equal probability, the conditional probability $\Pr(M_i|C_j)$ is simply $\Pr(M_i)$.

Example

- Plaintext: HELLO
- Keystream: EZABD
- Ciphertext: LDLMR

- Note that given the ciphertext LDLMR the plaintext can be *any* 5-letter message.

Real One Time Pads used by spies in 1960s

Vernam (binary) One Time Pad

- Plaintext is binary sequence: $b_1, b_2, \ldots, b_r$
- Keystream is random binary sequence: $k_1, k_2, \ldots, k_r$
- Ciphertext is binary sequence: $c_1, c_2, \ldots, c_r$
- Encryption: $c_i \equiv p_i \oplus k_i$
- Decryption: $p_i \equiv c_i \oplus k_i$
- Keystream is same length as plaintext
- Provides perfect secrecy since any ciphertext is equally possible
- Encryption and decryption are identical processes

One Time Pad properties

- Shannon showed that any cipher with perfect secrecy *must* have as many keys as there are messages.
- In this sense the One Time Pad is the only unbreakable cipher.
- Practical usage is possible for pre-assigned communications between fixed parties.
- Main problem with One Time Pad as a general tool is how to deal with key management of completely random keys.
- Key generation, key transportation, key synchronization, key destruction are all problematic since the keys are so large.
- In Caesar cipher, the key length was one integer between $\{0, \ldots, 26\}$ (≈ 5 bits). Now, the key is the length of the message, for all possible messages.

Key management issues for One Time Pad

- How to generate completely random keys?
- How to transport random keys between sender and receiver?
- How to synchronize on usage of keys?
- How to destroy keys after use?

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Visual cryptography

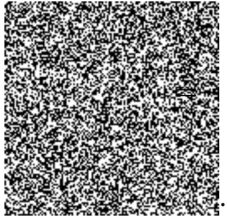
- A fun application of the One Time Pad is *visual cryptography* which splits an image into two *shares*
- Decryption works by *overlaying* the two shared images
- First proposed by Naor and Shamir in 1994
- We consider the simplest case of monochrome images with black or white pixels — many generalizations are possible
- Each share reveals no information about the image - this is unconditional security as in the One Time Pad

Two-time pad

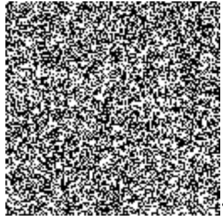
Message:

$\oplus$

OTP key:

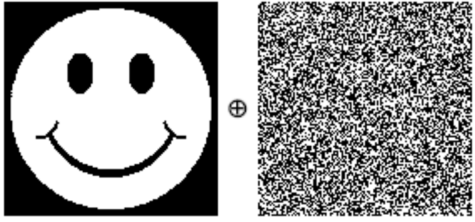


Ciphertext:

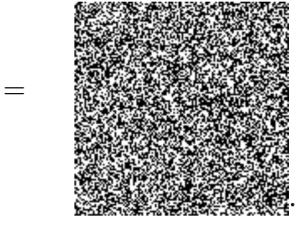


Two-time pad

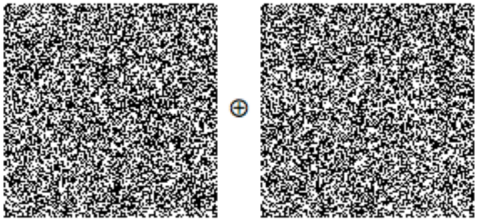
Message:



Ciphertext:



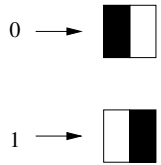
Two-time Pad



Two-time Pad

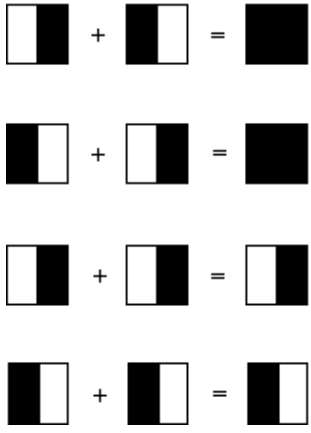
Encrypting in visual cryptography

- To encrypt an image I, first generate a One Time Pad P (random string of bits) with length equal to the number of pixels of I
- Generate an image share S_1 by replacing each bit in P using the sub-pixel patterns shown
- Generate the other image share S_2 with pixels as follows:
  - the same as S_1 for all the white pixels of I
  - the opposite (other sub-pixel pattern) of S_1 for all the black pixels of I



Decrypting in visual cryptography

- To reveal the hidden image the two shares are overlayed
- Each black pixel of *I* is black in the overlay
- Each white pixel of *I* is half white in the overlay



Questions?